Math 601 Midterm 1

Name: ______________________

This exam has 9 questions, for a total of 100 points.

Please answer each question in the space provided. You need to write **full solutions**. Answers without justification will not be graded. Cross out anything the grader should ignore and circle or box the final answer.

Question	Points	Score
1	18	
2	10	
3	15	
4	10	
5	10	
6	6	
7	10	
8	6	
9	15	
Total:	100	

Question 1. (18 pts)

Determine whether each of the following statements is true or false. You do NOT need to explain.

(a) Let U and W be subspaces of the vector space V. Then

$$\dim(U + W) = \dim U + \dim W.$$

Solution: False.

(b) An $(n \times n)$ matrix A is invertible if and only if the linear system $A\mathbf{x} = \mathbf{0}$ has a unique solution.

Solution: True.

(c) The rank of a matrix is equal to the dimension of its column space.

Solution: True.

(d) We can find an $(n \times n)$ matrix A such that A is invertible but A^T is not invertible.

Solution: False.

(e) For a homogeneous system of rank r and n unknowns, the dimension of the solution space is $n - r$.

Solution: True.

(f) Let $v_1, \cdots, v_k$ and w be vectors in a vector space V. Suppose $v_1, \cdots, v_k$ are linearly independent, then the set $\{w, v_1, \cdots, v_k\}$ is linearly independent if and only if w cannot be written as a linear combination of $v_1, \cdots, v_k$.

Solution: True.

Question 2. (10 pts)

Given two lines

$$L_1 : x = t + 1,\ y = 3t + 1,\ z = 2t - 1,$$

and

$$L_2 : x = 2t - 2,\ y = 2t + 3,\ z = t + 1,$$

suppose a plane H is parallel to both L_1 and L_2. Moreover, H passes through the point $(0, 1, 0)$. Find an equation of H.

Solution: The direction vector of L is orthogonal to both the normal vector $u = (1, 3, 2)$ of the plane and the direction vector $v = (2, 2, 1)$ of the other line.

Calculate the cross product of u and v:

$$u \times v = (-1, 3, -4)$$

So an equation of H is

$$-x + 3y - 4z = 3$$

Question 3. (15 pts)
Given

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix}$$

(a) Find a basis of Ker(A).

Solution: First, use elementary row operations to get the reduced row echelon form of A.

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So all elements in Ker$A$ are of the form

$$t \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So

$$v_1 = \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

form a basis of the kernel.

(b) Find a basis of the row space of A.

Solution: The three nonzero rows in the reduced row echelon form of A form a basis of the row space of A. That is

$$u_1 = (1, 2, 0, 5, 0)$$

$$u_2 = (0, 0, 1, -1, 0)$$

$$u_3 = (0, 0, 0, 0, 1)$$

form a basis of the row space of A.

(c) Find a basis of $\mathrm{Im}(A)$.

Solution: Use rref(A) from the part (a), we see that the 1st, 3rd and 5th columns of A form a basis of $\mathrm{Im}(A)$. That is,

$$w_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, w_2 = \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}, w_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

form a basis of $\mathrm{Im}(A)$.

Question 4. (10 pts)

Determine whether

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix} \text{ and } v_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix}$$

are linearly independent or not.

Solution: Form the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 4 \\ 1 & 3 & 7 & -1 \\ 1 & 4 & 10 & 2 \end{bmatrix}$$

by applying elementary row operations, we get an echelon form

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank is 3, which is < 4. In other words, v_1, v_2, v_3 and v_4 are linearly dependent.

Question 5. (10 pts)

Let $M_2(\mathbb{R})$ be the space of all (2×2) matrices with real coefficients. The set

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

is a basis of $M_2(\mathbb{R})$. Find the coordinates of $A = \begin{pmatrix} 2 & 3 \\ 4 & -7 \end{pmatrix}$ with respect to the basis S.

Solution: We need to write

$$\begin{pmatrix} 2 & 3 \\ 4 & -7 \end{pmatrix} = a_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + a_4 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

That is, we need to solve the linear system

$$\begin{cases} a_1 + a_2 + a_3 + a_4 = 2 \\ a_1 - a_2 - a_3 = 3 \\ a_1 + a_2 = 4 \\ a_1 = -7 \end{cases}$$

Simply use back substitution. We have

$$a_1 = -7, a_2 = 11, a_3 = -21, a_4 = 19$$

So

$$[A]_S = \begin{bmatrix} -7 \\ 11 \\ -21 \\ 19 \end{bmatrix}$$

Question 6. (6 pts)

Let A be an $(n \times m)$ matrix. Show that the solution space of $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^m$.

Solution: Denote the solution space by W.

(1) $0 \in W$, since $A\mathbf{0} = \mathbf{0}$,

(2) If $v, w \in W$, then

$$A(v + w) = A(v) + A(w) = 0 + 0 = 0.$$

So $v + w \in W$.

(3) If $w \in W$ and $k \in \mathbb{R}$, then

$$A(kw) = kA(w) = k \cdot 0 = 0.$$

So $kw \in W$.

Therefore W is a subspace of $\mathbb{R}^m$.

Question 7. (10 pts)

Show that $(2t-1)$, $(t+3)$ and $(t+1)^2$ form a basis of $\mathbb{P}_2(t)$, where $\mathbb{P}_2(t)$ is the space of all polynomials of degree ≤ 2.

Solution: There are various ways to solve this problem.

(1) First method: prove that $(t-1)$, $(t+1)$ and $(t-1)^2$ are linearly independent and span $\mathbb{P}_2(t)$.

(2) Second method: prove that $(t-1)$, $(t+1)$ and $(t-1)^2$ are linearly independent and use the fact $\dim \mathbb{P}_2(t) = 3$.

(3) third method: prove that $(t-1)$, $(t+1)$ and $(t-1)^2$ span $\mathbb{P}_2(t)$ and use the fact $\dim \mathbb{P}_2(t) = 3$.

Let us the second method. We know that $S = \{1, t, t^2\}$ is a basis of $\mathbb{P}_2(t)$. Write down the coordinate vectors of $(2t-1)$, $(t+3)$ and $(t+1)^2$ with respect to S.

$$v_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

It suffices to show that v_1, v_2 and v_3 are linearly independent. Form the matrix

$$\begin{bmatrix} -1 & 3 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

It echelon form is

$$\begin{bmatrix} -1 & 3 & 1 \\ 0 & 7 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

which has rank $= 3$. So v_1, v_2 and v_3 are linearly independent

Now we know that $\dim \mathbb{P}_2(t) = 3$. Then any 3 linearly independent vectors of $\mathbb{P}_2(t)$ form a basis. Therefore $(2t-1)$, $(t+3)$ and $(t+1)^2$ form a basis.

Question 8. (6 pts)

Let A be an (2×2) matrix. If we write

$$A = \begin{bmatrix} v & w \end{bmatrix}$$

where v and w are the columns of A, then we know that

$$A^T = \begin{bmatrix} v^T \\ w^T \end{bmatrix}$$

where v^T and w^T are the rows of A^T. Use this fact to show that if A is orthogonal, then v and w are orthogonal to each other.

Solution: By definition, A is orthogonal if $AA^T = A^T A = I$.

We have

$$A^T A = \begin{bmatrix} v^T \\ w^T \end{bmatrix} \begin{bmatrix} v & w \end{bmatrix} = \begin{bmatrix} v^T v & v^T w \\ w^T v & w^T w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Notice that $v^T w$ is precisely the dot product of v with w. Therefore, if A is orthogonal, then $v \cdot w = v^T w = 0$. So v and w are orthogonal.

Question 9. (15 pts)

The set of vectors $S = \{v_1, v_2, v_3\}$ with $v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, v_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, v_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$ is a basis of $\mathbb{R}^3$.

Also, the set of vectors $T = \{w_1, w_2, w_3\}$ with

$$w_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}, w_2 = \begin{bmatrix}2\\3\\4\end{bmatrix}, w_3 = \begin{bmatrix}1\\1\\2\end{bmatrix}$$

is another basis of $\mathbb{R}^3$. Accept these facts.

(a) Find the coordinates of w_1 with respect to S. Do the same for w_2 and w_3, that is, find the coordinates of w_2 and w_3 with respect to S.

Solution: Need to solve for a_1, a_2 and a_3 in

$$a_1v_1 + a_2v_2 + a_3v_3 = w_1$$

equivalently, solve the system

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

We get $a_3 = 3$, $a_2 = -1$ and $a_1 = -1$. So $[w_1]_S = \begin{bmatrix}-1\\-1\\3\end{bmatrix}$.

Similarly, we have

$$[w_2]_S = \begin{bmatrix}-1\\-1\\4\end{bmatrix}, [w_3]_S = \begin{bmatrix}0\\-1\\2\end{bmatrix}$$

(b) Take the coordinate vectors (with respect to S) of w_1, w_2 and w_3 from Part (a), and use these coordinate vectors as column vectors to form a matrix A. Determine whether A is invertible.

Solution:

$$A = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & -1 \\ 3 & 4 & 2 \end{bmatrix}$$

Its reduced echelon form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So A is invertible.